Journal of Statistical Physics, Vol. 53, Nos. 1/2, 1988

Correlated Random Walks

S. M. T. de la Selva,[1,2,4] Katja Lindenberg,[1,2] and Bruce J. West[2,3]

Recieved May 17, 1988

We present a new approach to the calculation of first passage statistics for correlated random walks on one-dimensional discrete systems. The processes may be non-Markovian and also nonstationary. A number of examples are used to demonstrate the theory.

KEY WORDS: Random walks; correlated; multistate; nonstationary; sluggish walker.

1. INTRODUCTION

Correlated walks on discrete lattices are useful in describing a number of physical phenomena.[1] The correlations arise when the direction of a step taken by the walker depends on the direction of the immediately preceding step. In an ordinary random walk these steps are independent. Examples of physical phenomena where such correlations may be important include polymer growth by sequential addition and deletion of single monomers: if growth occurs when there is a local monomer excess, then the addition of one monomer is likely to be followed by the further addition of another. Similarly, if a monomer detaches from the chain where there is a local

[1] Department of Chemistry, B-040, University of California at San Diego, La Jolla, California 92093.
[2] Institute for Nonlinear Science, R-002, University of California at San Diego, La Jolla, California 92093.
[3] Division of Applied Nonlinear Problems, La Jolla Institute, La Jolla, California 92038.
[4] Permanent address: Universidad Autónoma Metropolitana, Departamento de Física, Iztapalapa, Mexico 13 D.F.

0022-4715/88/1000-0203$06.00/0

monomer deficit, then there is a tendency for another monomer to detach. Another example involves a population with births and deaths: a net growth might indicate a healthy population in which the trend for subsequent growth follows as a manifestation of continued health. A net decrease might indicate an unhealthful population whose tendency is then to continue decreasing. A third example is biased flow through a branched structure: the walker must continue walking in a specified direction until it reaches a vertex. At that point a new direction can be chosen but must be retained until the walker encounters the next vertex. Finally, we may simply be interested in a random walker who has a tendency to continue walking in a given direction or a tendency to reverse directions.

A number of approaches have been developed recently to deal with correlated walks.[2,3] Herein we present a new point of view which has a number of advantages over other methods. Our approach generalizes methods developed by us for continuous processes[4,5] and is able to handle non-Markovian multivalued noise. We are particularly interested in the calculation of the statistics of first arrival of the walker at a prescribed site N.[6] In the polymer example, N represents a given length at which a particular event occurs (e.g., detachment from a surface). In the population example a size N might necessitate the design of new food supplies. In the case of the biased flow N may represent the edge of a fluidized bed.

In Section 2 we discuss a correlated nearest neighbor random walk and find that the mean first passage time out of an interval depends quadratically on the length of the interval [see Eq. (2.24)]. In Section 3 we consider a unidirectional walker who can either move in one direction or stand still. Our walker becomes more "sluggish" as it moves, and we find a mean first passage time out of an interval that may depend algebraically [Eq. (3.29)] or even exponentially [Eq. (3.21)] on the interval length, depending on the slowing-down tendencies of the walker. In Section 4 we present a prescription for dealing with non-nearest neighbor correlated walks. We end with a short discussion in Section 5.

2. CORRELATED NEAREST-NEIGHBOR RANDOM WALK

Consider a random walker on a chain of sites denoted by the integer index l. The walker can take steps to the right or to the left and can also remain at a given site. The steps can be of length 0, 1, 2,..., m; steps to the right (left) are labeled by positive (negative) integers. The probability that n consecutive steps of type k [i.e., in direction $\mathrm{sgn}(k)$ and of length $|k|$] are taken is $\psi_k(n)$. In an ordinary random walk $k=1$ or -1 and $\psi_k(n)=1/2$. In a correlated walk of the familiar sort[2,3] $k=1$ or -1 and $\psi_k(n)=(1-\gamma)\gamma^{n-1}$ for $n \geqslant 1$ with $\gamma>1/2$ ($\gamma<1/2$) if there is a tendency to retain (reverse) a given direction. We can deal with multiple values of $|k|$ and with more general (even history-dependent) forms of $\psi_k(n)$, i.e., the walk need not be a renewal process (see Section 3).

We concentrate on the calculation of the statistics of first arrival or crossing of a particular site $l=N$.[6] For some applications arrival at or crossing of two sites (e.g., $l=0$ or $l=N$) is of interest, and we shall deal with this case as well. For example, a population becoming extinct or a population reaching a critical size each corresponds to crossing of a given site. On the other hand, a random walk on a chain with traps at either end is a two-site crossing problem.

We introduce in this section a method that is particularly suited to walks having two possible values of k, e.g., $k=\pm 1$ or $k=0, 1$. This encompasses ordinary and correlated nearest neighbor random walks and is a discrete analogue of the continuous dichotomous noise process discussed elsewhere.[4] Walks with multiple values of k (i.e., an analogue of our earlier multistate continuous process[5]) are considered in Section 4.

Consider, then, a nearest neighbor random walker on a chain and suppose the walk starts at l_0 with $-M<l_0<N$. The position of the walker after n steps is related to its position after $n-1$ steps via the stochastic difference equation

$$l_n = l_{n-1} + g_n \tag{2.1}$$

where g_n is a random variable that can take on the values ± 1. In an ordinary random walk the subsequent value of g is unrelated to its previous value, i.e., g_n is independent of g_{n-1}. The probability of a "run" of p consecutive equal realizations of g (e.g., the probability of an ordinary random walker taking p consecutive steps to the right) is $\psi_{\pm 1}(p)=(1/2)^p$. In a correlated random walk this probability is replaced by an arbitrary one $\psi_k(p)$ that can build in a tendency to continue walking in a given direction or to reverse direction.

Suppose that the walker begins its walk toward the right and takes n_1 steps in that direction, i.e., $g_1 = g_2 = \cdots = g_{n_1} = 1$. It then reverses directions and takes n_2 steps to the left. A further reversal occurs and it again walks to the right, this time n_3 steps, and so on. The position of the walker after n steps is then as follows:

$$
l_n = \begin{cases}
l_0 + n, \\
\quad 0 \leqslant n \leqslant n_1 \\
l_0 + n_1 - (n - n_1), \\
\quad n_1 \leqslant n \leqslant n_1 + n_2 \\
l_0 + n_1 - n_2 + (n - n_1 - n_2), \\
\quad n_1 + n_2 \leqslant n \leqslant n_1 + n_2 + n_3 \\
l_0 + n_1 - n_2 + n_3 - (n - n_1 - n_2 - n_3), \\
\quad n_1 + n_2 + n_3 \leqslant n \leqslant n_1 + n_2 + n_3 + n_4 \\
\vdots \\
l_0 + n_1 - n_2 \cdots \pm n_{j-1} \mp (n - n_1 - \cdots - n_{j-1}), \\
\quad n_1 + \cdots + n_{j-1} \leqslant n \leqslant n_1 + \cdots + n_j
\end{cases} \tag{2.2}
$$

where the upper (lower) sign in the last general expression is for j odd (even). The time intervals n_i are random variables governed by the probabilities $\psi_1(n_i)$ for odd i and $\psi_{-1}(n_i)$ for even i.

We wish to calculate the probability $f_N(n \mid l_0)$ that the walker first exits the interval $[-M+1, N-1]$ through N and that it does so on the nth step. Together with the corresponding probability $f_{-M}(n \mid l_0)$ for exiting at $-M$, we can then achieve our ultimate goal of calculating the mean number of steps $\bar{n}$ (mean first passage time) for the walker to be trapped at sinks located at N and $-M$:

$$\bar{n} = \sum_{n=0}^{\infty} n[f_N(n \mid l_0) + f_{-M}(n \mid l_0)] \tag{2.3}$$

(for a semi-infinite interval we can set M or N to infinity). To construct $f_N(n \mid l_0)$, we call each sequence of equal values of g_n an "interval" and we define the auxiliary probability $p_j(n \mid l_0)$ to be the probability that first passage to N or $-M$ occurs during the jth interval on the nth step. Clearly,

$$f_N(n \mid l_0) = \sum_{j=0}^{\infty} p_{2j+1}(n \mid l_0) \tag{2.4}$$

while f_{-M} is the sum over even interval index. The probabilities $p_j(n \mid l_0)$ can be constructed directly from the possible trajectories of the walker. To see how this is done, consider the explicit trajectory (2.2). During the first interval the walker will not step on site N if $l_0 + n_1 < N$. The probability for this to be the case is

$$\mathrm{Prob}(n_1 < N - l_0) = \sum_{n_1=1}^{N-l_0-1} \psi_1(n_1) \tag{2.5}$$

Not crossing $-M$ during the second interval depends on the condition $l_0 + n_1 - n_2 > -M$, which in turn has the probability

$$\mathrm{Prob}(n_2 < M + l_0 + n_1) = \sum_{n_2=1}^{M+l_0+n_1-1} \psi_{-1}(n_2) \tag{2.6}$$

of being true. The pattern for writing the probability that there is no crossing in subsequent intervals up to the $2j$th should be clear. Crossing of $l = N$ during the $(2j+1)$th interval is assured if $l_{n_{2j+1}} \geqslant N$, and the probability for this event to occur is

$$\mathrm{Prob}(n_{2j+1} \geqslant N - l_0 - n_1 + n_2 - \cdots + n_{2j}) = \sum_{n_{2j+1} = N - l_0 - n_1 + \cdots + n_{2j}}^{\infty} \psi_1(n_{2j+1}) \tag{2.7}$$

Trapping on the nth step of the $(2j+1)$th interval requires that

$$l_n = l_0 + n_1 - n_2 + \cdots - n_{2j} + \Delta_{2j} = N \tag{2.8}$$

where

$$\Delta_j \equiv n - (n_1 + n_2 + \cdots + n_j) \tag{2.9}$$

Combining these step-by-step probabilities leads to the expression

$$\begin{aligned} p_{2j+1}(n \mid l_0) = & \sum_{n_1=1}^{N-l_0-1} \psi_1(n_1) \sum_{n_2=1}^{N+l_0+n_1-1} \psi_{-1}(n_2) \sum_{n_3=1}^{N-l_0-n_1+n_2-1} \psi_1(n_3) \\ & \times \cdots \sum_{n_{2j}=1}^{N+l_0+n_1-n_2+\cdots+n_{2j-1}-1} \psi_{-1}(n_{2j}) \\ & \times \cdots \sum_{n_{2j+1}=N-l_0-n_1+\cdots+n_{2j}}^{\infty} \psi_1(n_{2j+1}) \end{aligned} \tag{2.10}$$

It is convenient to define a step-number generating function according to

$$\hat{p}_j(u \mid l_0) \equiv \sum_{n=0}^{\infty} u^n p_j(n \mid l_0) \tag{2.11}$$

The mean first passage time (2.3) that we seek is related to this generating function in a particularly simple way:

$$\bar{n} = \frac{d}{du} \sum_{j=1}^{\infty} \hat{p}_j(u \mid l_0) \bigg|_{u=1} \tag{2.12}$$

Inspection of the appropriate transform of (2.10) leads to the recursion relation

$$\hat{p}_{2j+1}(u \mid l_0) = \sum_{n_1=1}^{N-l_0-1} \psi_1(n_1) \sum_{n_2=1}^{N+l_0+n_1-1} \psi_{-1}(n_2)\, \hat{p}_{2j-1}(u \mid l_0) \tag{2.13a}$$

for $j \geqslant 1$, with

$$\hat{p}_1(u \mid l_0) = \sum_{n_1=N-l_0}^{\infty} \psi_1(n_1)\, u^{N-l_0} \tag{2.13b}$$

to begin the sequence. A sum of (2.13a) over j leads to a recursion relation for

$$K_N(u \mid l_0) \equiv \sum_{j=0}^{\infty} \hat{p}_{2j+1}(u \mid l_0) \tag{2.14}$$

A similar sequence of steps can be followed for trapping at $-M$, ending with a recursion relation for the sum

$$K_{-M}(u \mid l_0) = \sum_{j=1}^{\infty} \hat{P}_{2j}(u \mid l_0) \tag{2.15}$$

For trapping at N or $-M$ it is the sum of these two functions which is of interest,

$$K(u \mid l_0) \equiv K_N(u \mid l_0) + K_{-M}(u \mid l_0) \tag{2.16}$$

since (2.12) then implies that

$$\bar{n} = \frac{d}{du} K(u \mid l_0) \bigg|_{u=1} \tag{2.17}$$

The equation satisfied by $K(u \mid l_0)$ is

$$K(u \mid l_0) = \hat{p}_1(u \mid l_0) + \hat{p}_2(u \mid l_0) + \sum_{n_1=1}^{N-l_0-1} \sum_{n_2=1}^{N+l_0+n_1-1} \psi_1(n_1)\, \psi_{-1}(n_2)\, u^{n_1+n_2} K(u \mid l_0 + n_1 - n_2) \tag{2.18}$$

with $\hat{p}_1$ given by (2.13b) and

$$\hat{p}_2(u \mid l_0) = \sum_{n_1=1}^{N-l_0-l} \sum_{n_2=N+l_0+n_1}^{\infty} \psi_1(n_1)\,\psi_{-1}(n_2)\,u^{N+l_0+2n_1} \tag{2.19}$$

Solving (2.18) and inserting the solution in (2.17) formally solves the problem.

To carry the calculation further, we must assume a particular form for the persistence function $\psi_k(n)$. We choose the form for which the random variable g_n in Eq. (2.1) is a Markov process:

$$\psi_k(n) = (1-\gamma_k)\,\gamma_k^{n-1}, \qquad n \geqslant 1 \tag{2.20}$$

For simplicity we choose $\gamma_1 = \gamma_{-1} \equiv \gamma$, i.e., the walker has the same persistence to the right and to the left. As mentioned earlier, $\gamma = 1/2$ corresponds to an ordinary random walk. The form (2.20) in (2.18) leads to the solution

$$K(u \mid l_0) = \left\{ \frac{r^{M+l_0}(1-\mu\gamma/r)}{u(1-\gamma)} - \frac{r^{-M-l_0}(1-u\gamma r)}{u(1-\gamma)} - r^{l_0-N+1} + r^{N-l_0-1} \right\}$$
$$\times \left\{ (1-\gamma) \left[\frac{r^{M+N-1}}{1-u\gamma r} - \frac{r^{-M-N+1}}{1-u\gamma/r} \right] \right\}^{-1} \tag{2.21}$$

where

$$r = \frac{1}{2} \left\{ \left[\gamma u + \frac{1}{\gamma u} - \frac{(1-\gamma)^2}{\gamma} u \right] - \left[\left(\gamma u + \frac{1}{\gamma u} - \frac{(1-\gamma)^2 u}{\gamma} \right)^2 - 4 \right]^{1/2} \right\} \tag{2.22}$$

Recall that the evaluation of (2.21) assumed that the walker began its walk from l_0 toward the right. To restore the more natural initial symmetry according to which the first step is to the right or left with respective probabilities of 1/2 one should replace (2.17) with

$$\bar{n} = \frac{1}{2} \frac{d}{du} \left[K(u \mid l_0) + K(u \mid N-M-l_0) \right]_{u=1} \tag{2.23}$$

We finally obtain for the mean first passage time to N or $-M$ the result

$$\bar{n} = (N-l_0)(l_0+M)\frac{1-\gamma}{\gamma} + (N+M)\frac{2\gamma-1}{2\gamma} \tag{2.24}$$

This is precisely the discrete analogue of Eq. (26) of the first citation in ref. 4. The first (quadratic) term reduces to the usual random walk contribution when $\gamma = 1/2$:

$$\bar{n} \xrightarrow[\gamma=1/2]{} (N-l_0)(l_0+M) \tag{2.25}$$

The contribution of the quadratic term grows in importance with decreasing γ, i.e., when the walker has an enhanced tendency to continually change its direction of motion. When $\gamma = 1$, on the other hand, the motion is "ballistic" and hence linear in the length of the interval:

$$\bar{n} \xrightarrow[\gamma=1]{} \frac{N+M}{2} \tag{2.26}$$

In general the motion is a combination of diffusive and ballistic contributions whose relative importance depends on γ.

We end this section by noting that our method is not restricted to the form (2.20) for the persistence function, and it is in this sense that our approach is more general than previous ones. Other forms were considered explicitly in the continuum case.[4,5]

3. AN INCREASINGLY MORE SLUGGISH WALKER

Consider the following walk, in one way simpler and in another more complicated than the correlated walker of the previous section. As in a cromatography column or in an appropriate gradient, the walker can either proceed in only one direction (say, to the right) or sit still, i.e., $k = 0$ or 1. This aspect of the walk is simpler than that of Section 2. However, here we allow the persistence functions describing the number of steps (time units) that the walker moves or sits still to depend on the interval j. This nonstationary aspect is more complicated than the previous examples, and the walk ceases to be a renewal process. In particular, we will consider the case in which the walker becomes increasingly more sluggish. With the forms

$$\psi_{1j}(n) = (1 - \gamma_{1j})\, \gamma_{1j}^{n-1} \tag{3.1}$$

and

$$\psi_{0j}(n) = (1 - \gamma_{0j})\, \gamma_{0j}^{n-1} \tag{3.2}$$

($0 < \gamma_{ij} < 1$) this can be accomplished by having γ_{1j} increase with j (greater tendency to stop) and γ_{0j} decrease with j (greater tendency to remain still). A moving particle that picks up additional mass as it goes along might behave in this fashion.

With no loss of generality we set $l_0 \equiv 0$ and calculate the mean first passage time $\bar{n}$ to $l = N$. In particular, we will obtain the asymptotic N dependence of $\bar{n}$ for large N. We assume that the walker initially begins by walking (rather than sitting still) at $n = 0$ (this condition can be relaxed and does not affect our results). The position of the walker as a function of step number is then as follows:

$$l_n = \begin{cases} n, & \\ \quad 0 \leqslant n \leqslant n_1 & \\ n_1, & \\ \quad n_1 \leqslant n \leqslant n_1 + n_2 & \\ n_1 + (n - n_1 - n_2), & \\ \quad n_1 + n_2 \leqslant n \leqslant n_1 + n_2 + n_3 & \\ n_1 + n_3, & \\ \quad n_1 + n_2 + n_3 \leqslant n \leqslant n_1 + n_2 + n_3 + n_4 & \\ \vdots & \\ n_1 + n_3 + \cdots + n_{2j-1}, & \\ \quad n_1 + n_2 + \cdots + n_{2j-1} \leqslant n \leqslant n_1 + \cdots + n_{2j} & \\ n - n_2 - n_4 - \cdots - n_{2j}, & \\ \quad n_1 + n_2 + \cdots + n_{2j} \leqslant n \leqslant n_1 + n_2 + \cdots + n_{2j+1} & \end{cases} \tag{3.3}$$

The probability that the walker first encounters N on the nth step during the $(2j+1)$th interval is

$$\begin{aligned} p_{2j+1}(n) = {} & (1 - \delta_{j,0}) \sum_{n_1=1}^{N-1} \psi_{11}(n_1) \sum_{n_2=1}^{\infty} \psi_{02}(n_2) \sum_{n_3=1}^{N-n_1-1} \psi_{13}(n_3) \sum_{n_4=1}^{\infty} \psi_{04}(n_4) \\ & \cdots \times \sum_{n_{2j}=1}^{\infty} \psi_{0,2j}(n_{2j}) \sum_{n_{2j+1}=N-n_1-n_3-\cdots-n_{2j-1}}^{\infty} \\ & \times \psi_{1,2j+1}(n_{2j+1})\, \delta_{n-n_2-n_4-\cdots-n_{2j},N} \\ & + \delta_{j,0} \sum_{n_1=N}^{\infty} \psi_{11}(n_1)\, \delta_{n,N} \end{aligned} \tag{3.4}$$

The sums over n_{2i} extend over all possible values of these indices since the walker is allowed to sit still arbitrarily long. The Kronecker deltas inside the sums ensure that it is exactly the nth step that brings the walker to site N. The last (separate) term in (3.4) accounts for walkers that arrive at N during the first interval.

The walking and stationary intervals naturally separate in Eq. (3.4) and can thus be treated independently. Let us therefore define for $j \geqslant 1$

$$I_{2j}(n) \equiv \sum_{n_2=1}^{\infty} \psi_{02}(n_2) \sum_{n_4=1}^{\infty} \psi_{04}(n_4) \cdots \sum_{n_{2j}=1}^{\infty} \psi_{0,2j}(n_{2j})\, \delta_{n-n_2-\cdots-n_{2j},N} \tag{3.5}$$

and

$$\begin{aligned} M_{2j+1} \equiv & \sum_{n_1=1}^{N-1} \psi_{11}(n_1) \sum_{n_3=1}^{N-n_1-1} \psi_{13}(n_3) \cdots \sum_{n_{2j-1}=1}^{N-n_1-\cdots-n_{2j-3}-1} \psi_{1,2j-1}(n_{2j-1}) \\ & \times \cdots \sum_{n_{2j+1}=N-n_1-\cdots-n_{2j-1}}^{\infty} \psi_{1,2j+1}(n_{2j+1}) \end{aligned} \tag{3.6}$$

Note that j must be $<N$ and that $I_{2j}(n)$ depends on n, while M_{2j+1} does not. The mean first passage time to N in terms of these functions is given by

$$\begin{aligned} \bar{n} &= \sum_{j=0}^{N-1} \sum_{n=0}^{\infty} p_{2j+1}(n) \\ &= \sum_{j=1}^{N-1} M_{2j+1} \sum_{n=0}^{\infty} n I_{2j}(n) + N\gamma_{11}^{N-1} \end{aligned} \tag{3.7}$$

Introducing the generating function

$$\hat{I}_{2j}(u) = \sum_{n=N}^{\infty} u^n I_{2j}(n) \tag{3.8}$$

allows us to reexpress the mean first passage time as

$$\bar{n} = \sum_{j=1}^{N-1} M_{2j+1} \frac{d}{du} \hat{I}_{2j}(u)\bigg|_{u=1} + N\gamma_{11}^{N-1} \tag{3.9}$$

With the form (3.2) in (3.5) it is a simple matter to evaluate the generating function (3.8) and the derivative that appears in (3.9). We find

$$\hat{I}_{2j}(u) = u^{N+j} \prod_{i=1}^{j} \frac{(1-\gamma_{0,2i})}{(1-u\gamma_{0,2i})} \tag{3.10}$$

and

$$\frac{d}{du} \hat{I}_{2j}(u)\bigg|_{u=1} = N + j + \sum_{i=1}^{j} \frac{\gamma_{0,2i}}{1-\gamma_{0,2i}} \tag{3.11}$$

The evaluation of M_{2j+1} is somewhat more complicated and is most simply carried out in terms of the generating function

$$\tilde{M}_{2j+1}(y) \equiv \sum_{N=2}^{\infty} y^N M_{2j+1} \tag{3.12}$$

(the explicit N dependence of M_{2j+1} has not been indicated but should be understood). For this function we find

$$\tilde{M}_{2j+1}(y) = \frac{y^{j+1}}{1 - y\gamma_{1,2j+3}} \prod_{i=0}^{j} \frac{1-\gamma_{1,2i+1}}{1-y\gamma_{1,2i+1}} \tag{3.13}$$

Its inversion yields 0 for $j \geqslant N$ and, for $j < N$,

$$M_{2j+1} = \frac{(-1)^j}{\gamma_{1,2j+3}} \prod_{i=0}^{\infty} \frac{i-\gamma_{1,2i+1}}{\gamma_{1,2i+1}} \times \sum_{m=0}^{j+1} \left\{ \gamma_{1,2m+1}^{N-j} \left[\prod_{\substack{k=0 \\ k \neq m}}^{j+1} \left(\frac{1}{\gamma_{1,2m+1}} - \frac{1}{\gamma_{1,2k+1}} \right) \right]^{-1} \right\} \tag{3.14}$$

To implement the results (3.11) and (3.14), we must now assume a form for the γ_{0j} and the γ_{1j}. We consider two cases: one where the slow-down of the walker occurs very rapidly (exponentially) and the other where it occurs slowly (power law). For the former case we take

$$\gamma_{0,2j} = 1 - e^{-ja} \tag{3.15a}$$

$$\gamma_{1,2j+1} = e^{-bj} \tag{3.15b}$$

Thus, $\gamma_{0,2j} \to 1$ and $\gamma_{1,2j+1} \to 0$ with increasing j. The expression (3.11) is straightforward to evaluate:

$$\left. \frac{d}{du} \hat{I}_{2j}(u) \right|_{u=1} = N + \frac{e^{ja}-1}{1-e^{-a}} \tag{3.16}$$

The expression for M_{2j+1} is more complicated:

$$M_{2j+1} = \frac{\prod_{k=1}^{j} (1-e^{-kb})}{e^{-b(j+1)(j+2)/2}} \sum_{i=1}^{j+1} \left\{ e^{-Nb_i} \left[\prod_{\substack{k=1 \\ k \neq i}}^{j+1} (e^{b(k-i)} - 1) \right]^{-1} \right\} \tag{3.17}$$

We evaluate (3.17) for large b (very sluggish walker) and retain only the leading ($i = 1$) contribution in the sum. We easily find that

$$M_{2j+1} \approx e^{-(N-j-1)b} \tag{3.18}$$

Equations (3.16) and (3.18) in (3.9) and retention of leading terms finally yields for $e^{Na} \gg 1$

$$\bar{n} \approx N + \frac{e^{Na}}{e^{a}-1} \tag{3.19}$$

with corrections of $O(\bar{n}e^{-Nb})$ that tend to reduce $\bar{n}$. The first term in (3.21) is the number of walking steps that the walker *must* take to get to N and is hence a lower bound on $\bar{n}$. The second term is the contribution to $\bar{n}$ from the idle periods. The exponential N dependence of $\bar{n}$ should be noted.

Consider now a case where the slowdown of the walker is more gentle than in the previous example. Thus, in place of (3.15) we now take

$$\gamma_{0,2j} = 1 - a/j \tag{3.20a}$$

$$\gamma_{1,2j+1} = b/(j+1) \tag{3.20b}$$

Here $\gamma_{0,2j}$ still $\to 1$ and $\gamma_{1,2,j+1} \to 0$ with increasing j, but more slowly than in (3.15). The evaluation of (3.11) is again straightforward:

$$\frac{d}{du} \hat{I}_{2j}(u)\bigg|_{u=1} = N + \frac{j(j+1)}{2a} \tag{3.21}$$

and M_{2j+1} is again more complicated:

$$M_{2j+1} = (j+1)!\, b^{N-j-1} \prod_{k=1}^{j} \left(1 - \frac{b}{k}\right) \sum_{i=1}^{j+1} \frac{(-1)^{i-1}}{i^{N-j}(i-1)!\,(j-i+1)!} \tag{3.22}$$

The sum over i is dominated by the $i=1$ term for each j; for each j the error in taking only the leading term is of $O(2^{-N})$ smaller than the term retained. Thus, we write

$$M_{2j+1} \approx (j+1)\, b^{N-j-1} \prod_{k=1}^{j} (1 - b/k) \tag{3.23}$$

and consequently

$$\bar{n} \approx \sum_{j=1}^{N-1} \left[N + \frac{j(j+1)}{2a} \right] (j+1)\, b^{N-j-1} \prod_{k=1}^{j} \left(1 - \frac{b}{k}\right) + Nb^{n-1} \tag{3.24}$$

To further estimate this result explicitly, let us consider the case of small b as an example. The product over k can be approximated by taking logarithms and reexponentiating:

$$\ln \prod_{k=1}^{j} \left(1 - \frac{b}{k}\right) \approx -b \sum_{k=1}^{j} \frac{1}{k} \approx -b(C + \ln j) \tag{3.25}$$

where $C = 0.57722...$ is Euler's constant,[7] and hence

$$\prod_{k=1}^{j} \left(1 - \frac{b}{k}\right) \approx \frac{e^{Cb}}{j^b} \approx \frac{1}{j^b} \tag{3.26}$$

With this form it is easy to see that the largest value of j dominates the sum (3.24). Retaining only the leading contribution to the $j = N - 1$ term finally gives

$$\bar{n} \approx N^{3-b}/2a \tag{3.27}$$

In contrast with the exponential N dependence of the more sluggish walker (3.19), the dependence here is algebraic, albeit with a greater power than the linear one for a unidirectional walker who does not slow down.[5]

4. MULTISTATE CORRELATED WALKS

An alternate formulation that easily allows for the inclusion of multiple values of k (see the Introduction) is based on functions $S_k(l, n)$ defined as follows[5]:

> $S_k(l, n) =$ probability that the walker has not exited $[-M+1, N-1]$ on or before the nth step, that a switch to a walk of type k occurred immediately after the nth step, and that the nth step took the walker to site l

These functions satisfy the set of equations

$$\begin{aligned} S_k(l, n) = \beta_k\, \delta_{l,l_0}\, \delta_{n,0} + \sum_{q=-m}^{m} \beta_{kq} \sum_{l'=-M+1}^{N-1} \sum_{n'=0}^{n-1} S_q(l', n') \\ \times \delta_{l-l'-q(n-n'),0}\, \psi_q(n-n') \end{aligned} \tag{4.1}$$

for $-M+1 \leqslant l \leqslant N-1$ and $-m \leqslant k \leqslant m$. Here β_k is the probability that the walker begins its walk in the direction $\mathrm{sgn}(k)$ with steps of length $|k|$, and we have assumed that a switch to this direction and step length occurred immediately after $n = 0$ (other initial conditions can also be considered). The coefficient β_{kq} is the probability of a switch from a walk of type q to one of type k, and $\beta_{kk} \equiv 0$. This formulation is restricted to renewal processes, i.e., $\psi_q(n)$ does not depend on previous history.

The Kronecker delta in Eq. (4.1) can be used to perform one of the sums. It is convenient to perform the sum over n' for $q \neq 0$ and the one over l' for $q = 0$:

$$S_k(l, n) = \beta_k \,\delta_{l,l_0}\, \delta_{n,0} + \sum_{\substack{q=-m \\ q \neq 0}}^{m} \beta_{kq} \sum_{l'=-M+1}^{N-1} S_q\left(l', n - \frac{l-l'}{q}\right) \times \psi_q\left(\frac{l-l'}{q}\right) \theta\left(\frac{l-l'}{q}\right) + \beta_{k0} \sum_{n'=0}^{n-1} S_0(l, n')\, \psi_0(n-n') \quad (4.2)$$

Here $\theta(x)$ is the Heaviside function, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leqslant 0$, and we have not indicated the l_0 dependence explicitly. It is convenient to define the generating function for any function of the step number,

$$\hat{F}(u) \equiv \sum_{n=0}^{\infty} u^n F(n) \quad (4.3)$$

as in Eq. (2.11). Multiplying (4.2) by u^n and summing over n then yields

$$\hat{S}_k(l, u) = \beta_k \,\delta_{l,l_0} + \sum_{\substack{q=-m \\ q \neq 0}}^{m} \beta_{kq} \sum_{l'=-M+1}^{N-1} \theta\left(\frac{l-l'}{q}\right) \times \psi_q\left(\frac{l-l'}{q}\right) \hat{S}_q(l', u)\, u^{(l-l')/q} + \beta_{k0}[\hat{\psi}_0(u) - \psi_0(0)]\, \hat{S}_0(l, u) \quad (4.4)$$

where we have used the convention $S_q(l, n) \equiv 0$ for $n < 0$.

To express the probability $f_N(n \mid l_0)$ introduced in Section 2 in terms of the $S_k(l, n)$ requires the probability $\psi_k(n)$ that a walk of type k persists for at least n steps:

$$\psi_k(n) = \sum_{n'=n}^{\infty} \psi_k(n') \quad (4.5)$$

In terms of these quantities,

$$f_N(n \mid l_0) = \sum_{k=1}^{m} \sum_{l=-M+1}^{N-1} S_k\left(l, n - \left(\frac{N-l}{k}\right)\right) \psi_k\left(\frac{N-l}{k}\right) \quad (4.6)$$

The generating function for f_N satisfies the relation

$$\hat{f}_N(u \mid l_0) = \sum_{k=1}^{m} \sum_{l=-M+1}^{N-1} \hat{S}_k(l, u)\, u^{(N-l)/k}\, \psi_k\left(\frac{N-l}{k}\right) \tag{4.7}$$

Equation (4.7) together with (4.4) constitutes the formal solution of the problem [as does (4.6) with (4.2)].

As an example of the application of this formalism, consider again the form

$$\psi_k(n) = (1-\gamma_k)\,\gamma_k^{n-1} \qquad \text{for} \quad n \geqslant 1 \tag{4.8}$$

with $0 < \gamma_k < 1$, which is the analogue of the exponential form in the corresponding continuous-time problem. With this form (4.4) becomes

$$\begin{aligned} \hat{S}_k(l, u) = {} & \beta_k\,\delta_{l,l_0} + \sum_{q=-m}^{-1} \beta_{kq} \frac{1-\gamma_q}{\gamma_q} \sum_{l'=l+1}^{N-1} \hat{S}_q(l', u)(\gamma_q u)^{(l-l')/q} \\ & + \sum_{q=1}^{m} \beta_{kq} \frac{1-\gamma_q}{\gamma_q} \sum_{l'=-M+1}^{l-1} \hat{S}_q(l', u)(\gamma_q u)^{(l-l')/q} \\ & + \beta_{k0} \frac{1-\gamma_0}{1-\gamma_0 z}\, z\hat{S}_0(l, u) \end{aligned} \tag{4.9}$$

To solve (4.9), it is convenient to introduce $U_k(w, u)$, the generating function of $\hat{S}_k(l, u)$ with respect to l:

$$U_k(w, u) = \sum_{l=1}^{N-1} w^l \hat{S}_k(l, u) \tag{4.10}$$

(we have chosen $M = 0$ in this example). Multiplying (4.2) by w^l and summing over l leads to

$$\begin{aligned} U_k(w, u) = {} & \beta_k w^{l_0} + \sum_{q=-m}^{-1} \frac{\beta_{kq}(1-\gamma_q)/\gamma_q}{1-(\gamma_q u)^{1/q} w} \\ & \times [(\gamma_q u)^{1/q} w U_q((\gamma_q u)^{-1/q}, u) - U_q(w, u)] \\ & + \sum_{q=1}^{m} \frac{\beta_{kq}(1-\gamma_q)/\gamma_q}{1-(\gamma_q u)^{1/q} w} \\ & \times [(\gamma_q u)^{1/q} U_q(w, u) - [(\gamma_q u)^{1/q} w]^N U_q((\gamma_q u)^{-1/q}, u)] \\ & + \frac{\beta_{k0}(1-\gamma_0)\,u}{1-\gamma_0 u} U_0(w, u) \end{aligned} \tag{4.11}$$

This is a set of $2m+1$ linear equations easily solved using standard techniques. Appropriate inversion then leads to the desired first crossing distribution.

These equations can of course be specialized to the nearest neighbor walk considered in Section 2. With $k=1$ and -1 and $\gamma_1=\gamma_{-1}\equiv\gamma$ we obtain

$$U_1(w,u)=\frac{wl_0}{2}+\frac{(1-\gamma)/\gamma}{1-w/\gamma u}\left[\frac{w}{\gamma u}U_{-1}(\gamma u,u)-U_{-1}(w,u)\right]$$

$$U_{-1}(w,u)=\frac{wl_0}{2}+\frac{(1-\gamma)/\gamma}{1-w\gamma u}\left[w\gamma uU_1(w,u)-(w\gamma u)^N U_1\left(\frac{1}{\gamma u},u\right)\right] \tag{4.12}$$

where we have set $\beta_{1,-1}=\beta_{-1,1}=1$ and $\beta_1=\beta_{-1}=1/2$. Solution of these simultaneous equations and inversion of the l-generating functions yields

$$\begin{aligned}\hat{S}_1(l,u)=&\frac{2\gamma-1}{2\gamma}\delta_{l,l_0}+\frac{\theta(l-l_0)}{2(r-r^{-1})}\left[\left(1-\frac{\gamma u}{r}\right)\left(\frac{2\gamma-1}{\gamma}-\frac{1}{r\gamma u}\right)r^{l+1-l_0}\right.\\&\left.-(1-\gamma ur)\left(\frac{2\gamma-1}{\gamma}-\frac{r}{\gamma u}\right)r^{l_0-l-1}\right]\\&+\frac{(1-\gamma)\,U_{-1}(\gamma u,u)}{(r-r^{-1})\,\gamma^2 u}\left[\left(1-\frac{\gamma u}{r}\right)r^l-(1-\gamma ur)\,r^{-l}\right]\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\hat{S}_{-1}(l,u)=&\frac{1}{2}\delta_{l,l_0}+\frac{\theta(l-l_0)}{2(r-r^{-1})}\left[\left(1-\frac{1}{\gamma ur}\right)\right.\\&\times\left(1+\frac{\gamma u}{r}\frac{2\gamma-1}{\gamma}\right)r^{l+1-l_0}\\&\left.-\left(1-\frac{r}{u\gamma}\right)\left(1+\gamma ur\frac{2\gamma-1}{\gamma}\right)r^{l_0-l-1}\right]\\&+\frac{(1-\gamma)^2}{\gamma^2(r-r^{-1})}+U_{-1}(\gamma u,u)(r^{l-1}-r^{-l+1})\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}U_{-1}(\gamma u,u)=&\gamma\left[\left(1-\frac{\gamma u}{r}\frac{2\gamma-1}{\gamma}\right)r^{N-l_0}-\left(1-\gamma ur\frac{2\gamma-1}{\gamma}\right)r^{-N+l_0}\right]\\&\times\left[2(1-\gamma)^2 u\left(\frac{r^{N-1}}{1-r\gamma u}-\frac{r^{-N+1}}{1-\gamma u/r}\right)\right]^{-1}\end{aligned} \tag{4.15}$$

with r given in Eq. (2.22).

The mean number of steps for first arrival at $l=0$ or $l=N$ is obtained from these quantities via Eq. (2.3). We find that

$$\hat{f}_N(u \mid l_0) + \hat{f}_0(u \mid l_0) = (u\gamma)^N \, U_1(1/u\gamma, u) + U_{-1}(u\gamma, u) \qquad (4.16)$$

[note that the average over the initial direction of the walk was here incorporated from the outset when we chose $\beta_1 = \beta_{-1} = 1/2$ in (4.11)]. A derivative of (4.16) with respect to u evaluated at $u=1$ yields the mean first passage time (2.24) with $M=0$.

5. CONCLUSION

We have presented two methods for dealing with correlated random walks on discrete lattices and, in particular, for calculating the mean first passage time $\bar{n}$ of such walkers out of a specified interval of length N. One method is based on an explicit classification of trajectories and allows us to deal with walks whose progress is described by a random variable that need not be a Markov process and that need not even by stationary. Thus, we can deal with walks in which the nature of the walk is history dependent. This method is practical only for nearest neighbor correlated walks. As examples we consider a nearest neighbor correlated walk of the usual form ($\bar{n} \sim N^2$), a unidirectional walk in which the walker "slows down" exponentially as the walk proceeds ($n \sim e^{Na}$), and one where the "slowing" is algebraic ($\bar{n} \sim N^{3-b}$, $b \ll 1$). The second method allows us to deal with non-nearest (but history-independent) correlated walks. We have restricted our examples to those we can handle analytically, but point out that these methods are well suited to numerical solution for more complicated walks.

ACKNOWLEDGMENT

This work was supported in part by NSF grant INT 85-13630 and NSF DMR grant 86-19650.

REFERENCES

1. S. Fujita, Y. Okamura, E. Blaisten, and S. V. Godoy, *J. Chem. Phys.* **73**:4569 (1980); Y. Okamura, E. Blaisten-Barojas, S. Fujita, and S. V. Godoy, *Phys. Rev. B* **22**:1638 (1980); S. Fujita, Y. Okamura, and J. T. Chen, *J. Chem. Phys.* **72**:3993 (1980); S. Fujita, E. Blaisten-Barojas, M. Torres, and S. V. Godoy, *J. Chem. Phys.* **75**:3097 (1981).
2. S. Goldstein, *Q. J. Mech. Appl. Math.* **4**:129 (1951); Y. Okamura, E. Blaisten-Barojas, S. V. Godoy, S. E. Ulloa, and S. Fujita, *J. Chem. Phys.* **76**:601 (1982); S. V. Godoy, *J. Chem. Phys.* **78**:6940 (1983); S. V. Godoy, F. Huelsz, and S. Fujita, *Acta Phys. Aust.* **55**:189 (1984).
3. J. W. Haus and K. W. Kehr, *Phys. Rep.* **150**:264 (1987).
4. J. Masoliver, K. Lindenberg, and B. J. West, *Phys. Rev. A* **33**:2177; **34**:1481, 2351 (1986).
5. G. H. Weiss, J. Masoliver, K. Lindenberg, and B. J. West, *Phys. Rev. A* **36**:1435 (1987).
6. G. H. Weiss, *Adv. Chem. Phys.* **13**:1 (1966).
7. L. B. W. Jolley, *Summation of Series* (Dover, New York, 1961), p. 15.